[21.0045]

$$u_n = \sum_{k=1}^{n} \frac{1}{k+n}$$

$$v_n = \sum_{k=n}^{2n} \frac{1}{k}$$

On a :

$$u_n = \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{n+n}$$

$$= \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$$

$$u_{n+1} = \frac{1}{1+n+1} + \frac{1}{2+n+1} + \frac{1}{3+n+1} + \dots + \frac{1}{(n-1)+n+1} + \frac{1}{n+n+1} + \frac{1}{(n+1)+n+1}$$

$$= \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

Donc $u_{n+1} - u_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{1+n}$

$$= \frac{1}{2n+1} + \frac{1}{2} \cdot \frac{1}{n+1} - \frac{1}{1+n}$$

$$= \frac{1}{2n+1} - \frac{1}{2} \frac{1}{n+1}$$

$$= \frac{(2n+2)-(2n+1)}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)}$$

On a donc $u_{n+1} - u_n \geq 0$

donc (u_n) est croissante.

de plus :

$$v_n = \sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$$

$$v_{n+1} = \sum_{k=n+1}^{2n+2} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}.$$

donc,

$$v_{n+1} - v_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n}$$

$$= \frac{(2n+2)n + (2n+1)n - (2n+1)(2n+2)}{(2n+1)(2n+2)n}$$

$$= \frac{\cancel{2n^2} + \cancel{2n} + \cancel{2n^2} + n - \cancel{4n^2} - 4n - \cancel{2n} - 2}{(2n+1)(2n+2)n}$$

$$= \frac{-3n-2}{(2n+1)(2n+2)n}$$

d'où $v_{n+1} - v_n \leq 0$

Donc (v_n) est décroissante.

Et enfin,

$$u_n - v_n = \sum_{k=1}^{n} \frac{1}{k+1} - \sum_{k=n}^{2n} \frac{1}{k}$$

$$= -\frac{1}{n}$$

donc $\lim\limits_{n \to +\infty} (u_n - v_n) = 0$

Les suites (u_n) et (v_n) sont donc adjacentes.