[10,0018]

1) I.P.P $u = \ln(t)$ donc $u' = \frac{1}{t}$

$v' = \frac{1}{t^2}$ on choisit $v = -\frac{1}{t}$

On peut alors écrire :

$$I_n = \left[-\frac{\ln(t)}{t}\right]_{e^n}^{e^{n+1}} - \int_{e^n}^{e^{n+1}} \left(-\frac{1}{t}\right) \cdot \frac{1}{t}\, dt$$

$$I_n = \left(-\frac{\ln(e^{n+1})}{e^{n+1}} + \frac{\ln(e^n)}{e^n}\right) + \int_{e^n}^{e^{n+1}} \frac{1}{t^2}\, dt$$

$$= -e^{-(n+1)} \cdot (n+1) \underbrace{\ln(e)}_{=1} + e^{-n} \cdot n \cdot \underbrace{\ln(e)}_{=1} + \left[-\frac{1}{t}\right]_{e^n}^{e^{n+1}}$$

$$= -e^{-(n+1)} \cdot (n+1) + e^{-n} \cdot n - e^{-(n+1)} + e^{-n}$$

$$= n \cdot e^{-n} + e^{-n} - (n+1) e^{-(n+1)} - e^{-(n+1)}$$

$$= (n+1) e^{-n} - (n+1+1) e^{-(n+1)}$$

$$I_n = \frac{n+1}{e^n} - \frac{n+2}{e^{n+1}}$$

2) $$A_n = \int_{e^n}^{e^{n+1}} f(x)\, dx = \int_{e^n}^{e^{n+1}} \frac{\ln(x) + xe}{x^2}\, dx = \underbrace{\int_{e^n}^{e^{n+1}} \frac{\ln(x)}{x^2}\, dx}_{= I_n} + e \cdot \int_{e^n}^{e^{n+1}} \frac{x}{x^2}\, dx$$

$$= I_n + e \cdot \int_{e^n}^{e^{n+1}} \frac{1}{x}\, dx$$

$$= I_n + e \cdot \left[\ln(x)\right]_{e^n}^{e^{n+1}}$$

$$= I_n + e \cdot \left(\ln(e^{n+1}) - \ln(e^n)\right)$$

$$= I_n + (n+1) \cdot e - n \cdot e$$

$$A_n = I_n + e$$

3) D'après la question 1, on a :

$$I_0 = \frac{1}{e^0} - \frac{2}{e^1} = 1 - 2e^{-1} = 1 - \frac{2}{e}$$

$$A_0 = I_0 + e = 1 - \frac{2}{e} + e$$

4) $\lim\limits_{n \to +\infty} I_n = 0$

donc $\lim\limits_{n \to +\infty} A_n = e$.