04.0034.C

1) a)

$$A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} \quad L_1 \leftarrow L_1 + L_2$$

$$= \begin{vmatrix} -2-\lambda & -2-\lambda & 0 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} \quad C_2 \leftarrow C_2 - C_1$$

$$= \begin{vmatrix} -2-\lambda & 0 & 0 \\ -4 & -2-\lambda & -3 \\ 3 & 0 & 1-\lambda \end{vmatrix}$$

$$= (-1)^{1+1} . (-2-\lambda) . (-2-\lambda)(1-\lambda)$$

$$P(\lambda) = (2+\lambda)^2 (1-\lambda)$$

Donc $\mathrm{Spec}(A) = \{-2, 1\}$

- $AX = -2X \iff \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \\ -2c \end{pmatrix}$

$$\iff \begin{cases} 2a + 4b + 3c = -2a \\ -4a - 6b - 3c = -2b \\ 3a + 3b + c = -2c \end{cases}$$

$$\iff \begin{cases} 4a + 4b + 3c = 0 \\ 4a + 4b + 3c = 0 \\ 3a + 3b + 3c = 0 \end{cases}$$

$\Leftrightarrow \begin{cases} a = -b \\ a + b + c = 0 \end{cases}$

$\Leftrightarrow \begin{cases} a = -b \\ c = 0 \end{cases}$

donc $E_{-2} = \text{Vect}\big((1,-1,0)\big)$

- $AX = X \Leftrightarrow \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\Leftrightarrow \begin{cases} 2a + 4b + 3c = a \\ -4a - 6b - 3c = b \\ 3a + 3b + c = c \end{cases}$

$\Leftrightarrow \begin{cases} a + 4b + 3c = 0 \\ 4a + 7b + 3c = 0 \\ a + b = 0 \end{cases}$

$\Leftrightarrow a = -b = c$

donc $E_1 = \text{Vect}\big((1,-1,1)\big)$

A n'est donc pas diagonalisable.

b) $B = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 4 & 1 \\ 0 & 1-\lambda & 2 & 3 \\ 0 & 0 & 3-\lambda & 3 \\ 0 & 0 & 3 & 3-\lambda \end{vmatrix}$$

$$= (-1)^{1+1} \cdot (2-\lambda) \cdot (-1)^{1+1} \cdot (1-\lambda) \cdot \left[(3-\lambda)^2 - 9\right]$$

$$= (2-\lambda)(1-\lambda)(3-\lambda-3)(3-\lambda+3)$$

$$P(\lambda) = -\lambda(1-\lambda)(2-\lambda)(6-\lambda)$$

Donc $\mathrm{Spec}(B) = \{0, 1, 2, 6\}$ → On peut donc affirmer dès maintenant que B est diagonalisable.

- $BX = 0 \iff \begin{cases} 2a + 4c + d = 0 \\ b + 2c + 3d = 0 \\ 3c + 3d = 0 \\ 3c + 3d = 0 \end{cases}$

$\iff \begin{cases} 2a = -3c \\ b = c = -d \end{cases}$

donc $E_0 = \mathrm{Vect}\left((3, -2, -2, 2)\right)$

- $BX = X \iff \begin{cases} 2a + 4c + d = a \\ b + 2c + 3d = b \\ 3c + 3d = c \\ 3c + 3d = d \end{cases}$

$$\Leftrightarrow \begin{cases} a + 4c + d = 0 \\ 2c + 3d = 0 \\ 3c + 2d = 0 \end{cases}$$

$$\Leftrightarrow \quad a = \quad c = d = 0$$

Donc $E_1 = \text{Vect}\left(\left(0, 1, 0, 0\right)\right)$

- $BX = 2x \Leftrightarrow \begin{cases} 2a + 4c + d = 2a \\ b + 2c + 3d = 2b \\ 3c + 3d = 2c \\ 3c + 3d = 2d \end{cases}$

$$\Leftrightarrow \begin{cases} 4c + d = 0 \\ -b + 2c + 3d = 0 \\ c + 3d = 0 \\ 3c + d = 0 \end{cases}$$

$$\Leftrightarrow b = c = d = 0$$

Donc $E_2 = \text{Vect}\left(\left(1, 0, 0, 0\right)\right)$

- $BX = 6X \Leftrightarrow \begin{cases} 2a + 4c + d = 6a \\ b + 2c + 3d = 6b \\ 3c + 3d = 6c \\ 3c + 3d = 6d \end{cases}$

$$\Leftrightarrow \begin{cases} -4a + 4c + d = 0 \\ -5b + 2c + 3d = 0 \\ -3c + 3d = 0 \\ 3c - 3d = 0 \end{cases}$$

$\Leftrightarrow \begin{cases} b = c = d \\ 4a = 5c \end{cases}$

Donc $E_6 = \mathrm{Vect}\left((5, 4, 4, 4)\right)$

Donc B est diagonalisable.

② a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4$$

$$P(\lambda) = (\lambda+1)(\lambda-4)$$

Donc $\mathrm{Spec}(A) = \{-1, 4\}$

- $AX = -X \Leftrightarrow \begin{cases} a + 2b = -a \\ 3a + 2b = -b \end{cases}$

$\Leftrightarrow \begin{cases} 2a + 2b = 0 \\ 3a + 3b = 0 \end{cases}$

$\Leftrightarrow a = -b$

Donc $E_1 = \mathrm{Vect}\left((1, -1)\right)$

- $AX = 4X \Leftrightarrow \begin{cases} a + 2b = 4a \\ 3a + 2b = 4b \end{cases}$

$\Leftrightarrow \begin{cases} -3a + 2b = 0 \\ 3a - 2b = 0 \end{cases}$

$\Leftrightarrow 3a = 2b$

Donc $E_4 = \text{Vect}\left((2,3)\right)$

Donc $D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

et $P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

donc $P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

et $A = P D P^{-1}$

donc $A^n = PDP^{-1} . PDP^{-1} . \dots . PDP^{-1}$

$= PD^n P^{-1}$

$= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 4^n \end{pmatrix} \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}$

$= \frac{1}{5} \begin{pmatrix} (-1)^n & 2.4^n \\ (-1)^{n+1} & 3.4^n \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

$A^n = \frac{1}{5} \begin{pmatrix} 3.(-1)^n + 2.4^n & 2.(-1)^{n+1} + 2.4^n \\ 3.(-1)^{n+1} + 3.4^n & 2.(-1)^n + 3.4^n \end{pmatrix}$

b) $B = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$

$P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$Spec(B) = \{1\}$

$E_1 = Vect((1,3))$

B n'est pas diagonalisable.

c) $C = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

$P(\lambda) = \lambda^2 - 3\lambda - 4$

$Spec(C) = \{-1, 4\} \rightarrow$ C est diagonalisable.

$E_4 = Vect((1,1))$

$E_{-1} = Vect((-3,2))$

$$C^n = \frac{1}{5} \begin{pmatrix} 3.(-1)^n + 2^{2n+1} & -3.((-1)^n - 4^n) \\ -2((-1)^n - 2^{2n}) & 2.(-1)^n + 3.4^n \end{pmatrix}$$

d) $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad P(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

$Spec(D) = \{0, 2\}$

$E_0 = Vect((-1,1))$ $\quad D^n = 2^{n-1}.D$.

$E_2 = Vect((1,1))$

3) a) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

$P(\lambda) = -\lambda^3 + 6\lambda^2 - 8\lambda + 3$

$\lambda_1 = \dfrac{5 + \sqrt{13}}{2}$

$\lambda_2 = 1$

$\lambda_3 = \dfrac{5 - \sqrt{13}}{2}$

Donc A est diagonalisable !

b) $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$P(\lambda) = -(\lambda - 1)^2 (\lambda - 4)$

$E_1 = \text{Vect}\left(\left(-1, 0, 1\right), \left(-1, 1, 0\right)\right)$

$E_4 = \text{Vect}\left(\left(1, 1, 1\right)\right)$

Donc B est diagonalisable.

c) $C = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$P(\lambda) = -(\lambda - 2)^2(\lambda - 1)$

$\mathrm{Spec}(C) = \{1, 2\}$

$E_2 = \mathrm{Vect}((-1, 1, 0))$

$E_1 = \mathrm{Vect}(((-1, 1, 1))$

C n'est pas diagonalisable

d) $D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$P(\lambda) = -\lambda^3 + 6\lambda - 9\lambda + 1$

$\lambda_1 \simeq 3,5$ → étude de fonction.

$\lambda_2 \simeq 2,3$

$\lambda_3 \simeq 0,1$

D est diagonalisable.