04.0009.C

① $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Recherche des valeurs propres

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \quad L_1 \leftarrow L_1 - L_3$$

$$= \begin{vmatrix} -2-\lambda & 0 & 2+\lambda \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \quad C_1 \leftarrow C_1 + C_3$$

$$= \begin{vmatrix} 0 & 0 & 2+\lambda \\ 2 & 3-\lambda & 1 \\ 4-\lambda & 1 & 1-\lambda \end{vmatrix}$$

$$= (-1)^{1+3} \cdot (2+\lambda) \cdot \left(2 - (3-\lambda)(4-\lambda)\right)$$

$$= (2+\lambda) \cdot \left(2 - 12 + 3\lambda + 4\lambda - \lambda^2\right)$$

$$= (2+\lambda) \cdot (-\lambda^2 + 7\lambda - 10)$$

$$P(\lambda) = -(2+\lambda)(\lambda-2)(\lambda-5)$$

Donc $\mathrm{Spec}(A) = \{-2, 2, 5\}$

② Recherche des vecteurs propres avec $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

- $AX = -2X \iff \begin{cases} a + b + 3c = -2a \\ a + 3b + c = -2b \\ 3a + b + c = -2c \end{cases}$

$\iff \begin{cases} 3a + b + 3c = 0 \\ a + 5b + c = 0 \\ 3a + b + 3c = 0 \end{cases}$

$\iff \begin{cases} 3a + b + 3c = 0 \\ a + 5b + c = 0 \end{cases}$

$\iff \begin{cases} 3(a+c) + b = 0 \\ a + c + 5b = 0 \end{cases}$

$\iff \begin{cases} -15b + b = 0 \\ a + c = -5b \end{cases}$

$\iff \begin{cases} b = 0 \\ a = -c \end{cases}$

Donc $E_{-2} = \text{Vect}\left((-1, 0, 1) \right)$

- $AX = 2X \iff \begin{cases} a + b + 3c = 2a \\ a + 3b + c = 2b \\ 3a + b + c = 2c \end{cases}$

$$\iff \begin{cases} a - b - 3c = 0 \\ a + b + c = 0 \\ 3a + b - c = 0 \end{cases}$$

$$\iff \begin{cases} 2a - 2c = 0 \\ a + b + c = 0 \\ 3a + b - c = 0 \end{cases}$$

$$\iff \begin{cases} a = c \\ b = -2a \end{cases}$$

Donc $E_2 = \mathrm{Vect}\left((1, -2, 1)\right)$

- $AX = 5X \iff \begin{cases} a + b + 3c = 5a \\ a + 3b + c = 5b \\ 3a + b + c = 5c \end{cases}$

$$\iff \begin{cases} 4a - b - 3c = 0 \\ a - 2b + c = 0 \\ 3a + b - 4c = 0 \end{cases}$$

$$\iff \begin{cases} 7a - 7c = 0 \\ a - 2b + c = 0 \end{cases}$$

$$\iff a = b = c$$

Donc $E_5 = \mathrm{Vect}\left((1, 1, 1)\right)$

On a,

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

et $P^{-1} = \frac{1}{6} \begin{pmatrix} -3 & 0 & 3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

et $A = PDP^{-1}$

donc

$$A^n = \frac{1}{6} \begin{pmatrix} 3.(-2)^n + 2^n + 2 \times 5^n & -2(2^n - 5^n) & -3(-2)^n + 2^n + 2 \times 5^n \\ -2(2^n - 5^n) & 2(2^{n+1} + 5^n) & -2(2^n - 5^n) \\ -3(-2)^n + 2^n + 2 \times 5^n & -2(2^n - 5^n) & 3(-2)^n + 2^n + 2 \times 5^n \end{pmatrix}$$

③ On pose $X_n = \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$

On a $X_{n+1} = A X_n$

On conjecture $X_n = A^n X_0$ avec $X_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$

<u>Recurrence</u>

=> Initialisation : $X_1 = A X_0$: ok

=> Hérédité : On suppose pour un n fixé $X_n = A^n X_0$.
Montrons que $X_{n+1} = A^{n+1} X_0$.

On a : $X_n = A^n X_0$.

avec $A X_n = X_{n+1}$ et $A A^n X_0 = A^{n+1} X_0$.

donc $X_{n+1} = A^{n+1} X_0$.

=> Conclusion : pour $n \in \mathbb{N}^*$, $X_n = A^n X_0$.