04.0005.C

① $A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$

$$P(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ -4 & -1-\lambda & 0 \\ 4 & -8 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda) \cdot \big((3-\lambda)(-1-\lambda) + 4\big)$$

$$= (2+\lambda)\big((\lambda-3)(-1-\lambda) - 4\big)$$

$$= (2+\lambda)\big(-\lambda - \lambda^2 + 3 + 3\lambda - 4\big)$$

$$= (2+\lambda)\big(-\lambda^2 + 2\lambda - 1\big)$$

$$= -(2+1)(\lambda-1)^2$$

Donc $\mathrm{Spec}(A) = \{-2, 1\}$.

$$AX = -2X \iff \begin{cases} 3a + b = -2a \\ -4a - b = -2b \\ 4a - 8b - 2c = -2c \end{cases}$$

$$\iff \begin{cases} 3a + 3b = 0 \\ 4a - b = 0 \\ 4a - 8b = 0 \end{cases}$$

$$\iff a = b = 0$$

$$E_{-2} = \mathrm{Vect}\big((0,0,1)\big)$$

$$AX = X \iff \begin{cases} 3a + b = a \\ -4a - b = b \\ 4a - 8b - 2c = c \end{cases}$$

$$\iff \begin{cases} 2a + b = 0 \\ 4a + 2b = 0 \\ 4a - 8b - 3c = 0 \end{cases}$$

$$\iff \begin{cases} b = -2a \\ 20a = 3c \end{cases}$$

Donc $E_1 = \text{Vect}((3, -6, 20))$

A n'est pas diagonalisable.

② $f(u_1) = -2u_1 \iff AU_1 = -2V_1$

$$\iff U_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$f(u_2) = u_2 \iff AU_2 = U_2$

$$\iff U_2 = \begin{pmatrix} 3 \\ -6 \\ 20 \end{pmatrix}$$

$f(u_3) = u_2 + u_3 \iff A V_3 = U_2 + U_3$

$$\Leftrightarrow \begin{cases} 3a + b = 3 + a \\ -4a - b = -6 + b \\ 4a - 8b - 2c = 20 + c \end{cases}$$

$$\Leftrightarrow \begin{cases} 2a + b = 3 \\ 4a - 8b - 3c = 20 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = 3 - 2a \\ 3c = 20a - 44 \end{cases}$$

Donc $U_3 = \begin{pmatrix} a \\ 3 - 2a \\ \frac{20}{3} a - \frac{44}{3} \end{pmatrix}$

on peut choisir $a = 1$

On a donc $U_3 = \left(1, 1, \frac{24}{3}\right)$

Donc A est semblable à la matrice à

$$T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

la matrice de passage étant

$$Q = \begin{pmatrix} 0 & 3 & 1 \\ 0 & -6 & 1 \\ 1 & 20 & \frac{24}{3} \end{pmatrix}$$

Il n'y a pas unicité, ni de la matrice triangulaire supérieure T, ni de la matrice de passage Q.